

Distributed Inter-Cell Interference Mitigation Via Joint Scheduling and Power Control Under Noise Rise Constraints

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Abstract—Consider the problem of joint uplink scheduling and power allocation. Being inherent to almost any wireless system, this resource allocation problem has received extensive attention. Yet, most common techniques either adopt classical power control, in which users are received with the same Signal to Interference-plus-Noise Ratio (SINR), or use centralized schemes, in which base stations coordinate their allocations. Modern systems such as IEEE 802.16e/m or 3GPP LTE, however, require inter-cell interference mitigation without compromising the benefits of distributed algorithms.

We suggest a novel scheduling approach in which each base station, besides allocating the time and frequency resources according to given constraints, also manages its uplink power budget such that the aggregate interference, “Noise Rise”, caused by its subscribers at the neighboring cells is bounded. Our suggested scheme is completely distributed between the cells, requiring neither coordination nor message exchange.

We rigorously define this allocation problem under noise rise constraints, give the optimal solution and derive an efficient iterative algorithm for it. We then discuss a relaxed problem, where the noise rise is constraint separately for each sub-channel or resource unit. While sub-optimal, this view renders the scheduling and power allocation problems separate, yielding an even simpler and more efficient solution, while the essence of the scheme is kept. Via extensive simulations, we show that the suggested approach increases overall throughput dramatically, with the same level of fairness and power consumption.

I. INTRODUCTION

The desire to provide integrated broadband services while maintaining *Quality of Service (QoS)* guarantees bestows growing interest in schedule access techniques used in multiple-access protocols for future broadband radio systems. Hence, for example, *IEEE 802.16e,m (WiMAX)* [1], [2] adopted *Orthogonal Frequency-Division Multiple Access (OFDMA)* as the access mechanism. The design of radio resource mechanisms, and in particular scheduling and power control, while maintaining some type of fairness, is challenging due to the high variability in link-quality. In this work, we consider the joint uplink scheduling and power control problem for wireless networks.

Common power-control approaches to the uplink resource allocation problem are either to assign transmission power to the mobile stations such that all users are received at the base station with the same *Signal to Interference-plus-Noise Ratio (SINR)* [3], [4], or to allow users to transmit at their maximal available power [5]. Both of these techniques optimize the

user/cell throughput, neglecting the interference injected to neighboring cells. However, since OFDMA systems, for example, are sensitive to inter-cell interference, the interference from neighboring cells can dramatically decrease the SINR received at the base station, hence reduce the mobile station throughput. As a result, the role of the power control becomes decisive in providing the required SINR while controlling at the same time the interference caused to neighboring cells. This is challenging, as on the one hand a mobile station near the base station is expected to have high quality link, hence high throughput, even when transmitting in low power, while a distant mobile needs to transmit in a much higher power to attain the same throughput, and on the other hand as far as inter-cell interference is concerned, mobiles near a base station can transmit in high power since they are not in the proximity of other cells, while distant mobiles which can be in the proximity of other cells should not transmit in high power as they can interfere with other (neighboring) base stations.

In order to limit the interference to neighboring cells, 3GPP has approved the use of *Fractional Power Control (FPC)* [6]. This new proposal makes users with a higher path-loss operate at a lower SINR requirement so that they will more likely generate less interference to neighboring cells. According to this approach, to preserve fairness, most of the resources should be allocated to far-away users. Similarly, the IEEE 802.16m power control scheme deducts a fraction of the downlink signal-to-interference ratio (SIR) from the transmission power. By that, it reduces the interference caused by cell-edge users. The 3GPP LTE standard also suggests a different approach for inter-cell interference handling termed *Inter-cell interference coordination (ICIC)* (see, e.g., [7]). ICIC provides tools for dynamic inter-cell-interference coordination of the scheduling in neighboring cells such that cell-edge users in different cells are preferably scheduled in complementary parts of the spectrum when required. However, ICIC requires coordination between neighboring cells both in terms of exchanging information regarding subscribers at one cell and their interference level on other neighboring cells, as well as coordination in the resource allocation, which further complicates the scheduling process.

In this work, our contribution is three-fold. First, we introduce a different approach, which controls the inter-cell interference, yet does not require any cross-deployment communication or coordination. In our approach, the uplink inter-

cell interference that each base station is allowed to create is bounded. This limited egress interference budget, termed Noise Rise, is then allocated to users in conjunction with the ordinary resources (time and frequency) allocation, according to some fairness criterion and channel condition. Controlling the average interference level sensed by each base station *provides a more predictable uplink SINR*, which allows a *lower interference margins and a more efficient rate selection*. Hence, it obtains a higher capacity and a better coverage.

Second, we formalize the scheduling problem under the noise rise constraint as a constraint convex optimization problem, and provide an efficient iterative algorithm that is proved to solve it optimally. Moreover, we suggest a second setting, in which instead of bounding the average noise rise over all the channels, allowing some sub-channels to contribute more noise rise at the expense of further limiting the noise rise on others, we bound the noise rise on *each sub-channel to the exact same value*. The later settings allows the decoupling of the scheduling algorithm from the power control and thus facilitates an even simpler algorithm.

Third, we thoroughly evaluate the noise rise concept via an extensive set of simulations, using both the direct analytical setting defined by the optimization problem and a more elaborate, all-inclusive setting defined by IMT-Advanced [8]. These simulations clearly depict that the suggested approach dramatically increases the overall throughput achieved in each cell compared to the traditional approach, while maintaining a high level of fairness.

II. NOISE RISE

Traditional cellular communications such as CDMA networks suffer from intra-cell interference caused by the pseudo orthogonality of the intra cell CDMA codes. Alternatively, the orthogonality of the sub-carriers in OFDMA networks solves the intra-cell interference resulting in inter-cell interference limitations. In such networks, the interference levels seen in a base station depends on the power control and scheduling of neighboring cells and not only on its own resource allocation. Accordingly, an optimal scheduling and power allocation would require coordination and synchronization across all deployment. Clearly, such a synchronization could be prohibitively complex and impractical. Without cross deployment coordination, where each cells performs its power control and scheduling independently of its neighbors, the interference profile seen in uplink transmission becomes highly dynamic. To overcome the variability of the inter-cell interference a high interference margin is accounted for while selecting the transmission rate and coding scheme. That is, transmitting in a more robust rate (lower rate) to maintain reliability at worse cases of the interference pattern. Clearly, a hight noise margin assumption reduces the cells uplink spectral efficiency.

In addition to the cell capacity, constrained interference is vital for cell coverage. If the noise plus interference is not constrained, due to the maximal power transmission of a mobile station, certain users (typically at the cell edge) would not be able to maintain a reasonable communication capacity.

These users would be blocked and removed from the coverage area, resulting in a reduced cell coverage.

Typically, the interference is represented as a measure of noise rise, which is defined as follows.

Definition 1. Uplink *noise rise* is defined as the total uplink received noise plus interference power over the background noise power. Formally, let γ denote the noise rise, then $N_0 + I = \gamma \cdot N_0$, where I is the total interference at the base station (receiver), and N_0 is the background noise.

The constraint noise rise approach, which is described in the next section, provides a fully distributed mechanism (without a cross deployment coordination and synchronization) that dramatically reduce the variability of the noise rise (due to inter-cell interference) allowing a more aggressive rate selection (taking a much smaller noise rise margin).

A. Constrained noise rise approach

The constrained noise rise approach aims at allowing the scheduler to operate in a fixed Noise plus Interference conditions and facilitating a more aggressive rate selection. Specifically, we aim at bounding the egress interference and show that it also bounds the noise rise seen by uplink transmission at all cells. Formally, let $\mathcal{M}(k)$ denote the set of backlogged users at base station k . Denote by $\mathcal{G}(k^*)$ the set of neighboring base stations of base station k^* and by $I_{in}(k^*)$ the ingress interference level per resource unit at base station k^* receiver,

$$I_{in}(k^*) = \sum_{k \in \mathcal{G}(k^*)} \sum_{i(k) \in \mathcal{M}(k)} L_{i(k),k^*} \cdot p_{i(k)} \quad (1)$$

where, $L_{i(k),k^*}$ denotes the channel gain between user $i(k)$ of base station k and base station k^* , and $p_{i(k)}$ the transmission power of user $i(k)$. In (1), the first summation is over all neighboring cells except the serving cell k^* , and the second summation is over all users in the neighboring cells.

On the other hand, the egress interference by the users of cell k^* to the neighbors is given by

$$I_{eg}(k^*) = \sum_{i(k^*) \in \mathcal{M}(k^*)} \sum_{k \in \mathcal{G}(k^*)} L_{i(k^*),k} \cdot p_{i(k^*)}. \quad (2)$$

Assuming a fully homogeneous and symmetric deployment, we have that on the average the ingress interference that neighboring cells injects to a base station equals to the egress interference caused by the base station to the neighboring cells. That is,

$$\begin{aligned}
& \mathbb{E}[I_{in}(k^*)] \\
&= \mathbb{E} \left[\sum_{k \in \mathcal{G}(k^*)} \sum_{i(k) \in \mathcal{M}(k)} L_{i(k),k^*} \cdot p_{i(k)} \right] \\
&= \sum_{k \in \mathcal{G}(k^*)} \mathbb{E} \left[\sum_{i(k) \in \mathcal{M}(k)} L_{i(k),k^*} \cdot p_{i(k)} \right] \\
&= |\mathcal{G}(k^*)| \cdot \mathbb{E} \left[\sum_{i(k) \in \mathcal{M}(k)} L_{i(k),k^*} \cdot p_{i(k)} \right] \\
&= \sum_{j \in \mathcal{G}(k)} \mathbb{E} \left[\sum_{i(k) \in \mathcal{M}(k)} L_{i(k),j} \cdot p_{i(k)} \right] \\
&= \mathbb{E} \left[\sum_{j \in \mathcal{G}(k)} \sum_{i(k) \in \mathcal{M}(k)} L_{i(k),j} \cdot p_{i(k)} \right] \\
&= \mathbb{E} \left[\sum_{i(k) \in \mathcal{M}(k)} \sum_{j \in \mathcal{G}(k)} L_{i(k),j} \cdot p_{i(k)} \right] \\
&= \mathbb{E}[I_{eg}(k)] \tag{3}
\end{aligned}$$

Therefore, maintaining a constrained egress interference at all deployment, realizes our goal of a constrained ingress interference.

Let l_i denote the normalized interference that a user i 's uplink transmission injects to its neighboring cells, that is

$$l_i = \sum_{k \in \mathcal{G}(k^*)} L_{i(k^*),k^*} \tag{4}$$

The interference caused by the users of cell k^* , $I_{eg}(k^*)$, can be written as

$$I_{eg}(k^*) = \sum_{i(k^*) \in \mathcal{M}(k^*)} l_{i(k^*)} \cdot p_{i(k^*)} \tag{5}$$

We are now ready to write the noise rise constraint on the scheduler and power control allocation. The constraint bounds the egress interference caused by the cell uplink transmission such that the ingress interference at all the deployment would remain bounded as well. The scheduler and power control should comply with the following constraint:

$$\sum_{i(k^*) \in \mathcal{M}(k^*)} l_{i(k^*)} \cdot p_{i(k^*)} \leq I \tag{6}$$

We note that (6) implies that the scheduler may consider noise rise as a resource to be allocated to transmitting users. That is, instead of considering the traditional power allocation, the noise rise approach considers noise rise allocation, i.e., $l_i \cdot p_i$. Accordingly, for example, if the egress interference of a given user is high, the scheduler should reduce its transmission power and increase its allocated bandwidth. Alternatively, the schedule could co-allocate this user with another user that "consumes" less noise-rise such that the noise rise budget is kept.

In the following sub-section we provide the means for a base station to estimate the required normalized interference.

B. Normalized interference

To comply with (6) each base station should have information on the normalized interference l_i . Typically, a base station cannot directly measure this coefficient. In fact, the normalized interference can be measured at the surrounding cells and reported to the base station (e.g., via Inter Cell Interference Coordination). Alternatively, a base station may estimate the normalized interference from its users' downlink channel state reports without the need for inter-cell communication or coordination.

The downlink SIR (signal-to-interference ratio) measured by a user is given by

$$SIR_{i(k^*)}^{DL} = \frac{L_{k^*,i(k^*)} P^{DL}}{\sum_{k \in \mathcal{G}(k^*)} L_{k,i(k^*)} P^{DL}} \tag{7}$$

where P^{DL} is the base station downlink transmission power. With the channel reciprocity [9], i.e., $L_{k,i} = L_{i,k}$, we have

$$SIR_{i(k^*)}^{DL} = \frac{L_{i(k^*),k^*}}{\sum_{k \in \mathcal{G}(k^*)} L_{i(k^*),k}} \tag{8}$$

Therefore,

$$l_i = \sum_{k \in \mathcal{G}(k^*)} L_{i(k^*),k} = \frac{L_{i(k^*),k^*}}{SIR_{i(k^*)}^{DL}} \tag{9}$$

Finally, we note that in an interference-limited scenario (where the interference is much larger than the background noise), which is typical, one could use the downlink SINR instead of the downlink SIR, neglecting the background noise.

III. SYSTEM MODEL

We consider the problem of scheduling and resource allocation for the uplink of an OFDMA cell, where a set $\mathcal{M} = \{1, \dots, M\}$ of users (with backlogged traffic) transmit to the same base station. The total frequency band is divided into a set $\mathcal{N} = \{1, \dots, N\}$ of *distributed resource allocation units* (e.g., frequency sub-bands or sub-channels). The distributed resource allocation units are of the same size as a physical resource units that has undergone sub-band partitioning and permutations. The permutation spreads the resource units' sub-carriers across the whole frequency band. Accordingly, for each user i the channel quality of all logical resource units can be assumed identical. The channel quality of a resource unit represents a collective quality indicator for the resource unit, e.g., the average across the sub-carriers in the unit.

Let x_i , $0 \leq x_i \leq 1$, be the fraction of the resource (i.e., fraction of the frequency band) allocated to user i . The total allocation across all users should be no larger than 1, i.e.

$$\sum_{i \in \mathcal{M}} x_i \leq 1 \tag{10}$$

In practical systems, only a single user can transmit over a given resource unit (assuming a single user MIMO), accordingly, x_i is a fraction of the total number of resource units. We consider systems with a large number of resource units where

x_i could take any fractional value neglecting the rounding errors (e.g., 92 resource units for a 20MHz band in IEEE 802.16m). Note that the rounding here differs from that of [5], which considers the fraction of a single resource unit (i.e., x_i could take only 0 or 1). Differently, we consider the fraction of the divisible whole frequency band.

Denote by p_i the transmission power of a user i over all resource units, which is subject to a maximum power level, $p_i < P$. In this paper we consider interference limited scenarios where the transmission power is bounded by the constrained noise rise rather than the mobile transmission power capabilities. Accordingly, in the following we shall assume that users do not have a maximal power constraint.

Time is divided into equal length time slots (i.e., sub-frames in the IEEE 802.16m terminology). At the beginning of every time slot, the scheduler seeks to maximize a (time-varying) weighted sum of the users rates.

The scheduler and power control mechanism are assumed to have knowledge of the channel gain, which comprises of the long term parameters of the link between the base station and the user, such as path loss and shadowing factor, as well as the short-term time-varying spatial fading. Let L_i be the channel gain from user i to the serving base station. Additionally, the scheduler is aware of the target Interference plus Noise at the serving base station, which translates to a normalized received signal-to-interference plus noise ratio (SINR) per unit transmit power, e_i , for each user i .

We assume that $r_i(t)$ is the achievable rate of user i at time t in a Gaussian Multiple Access Channel using timesharing [10]. Formally, $r_i(t)$ is given by

$$r_i(t) = x_i \log \left(1 + \frac{p_i e_i(t)}{x_i} \right). \quad (11)$$

where the tuple $(\mathbf{x}, \mathbf{p}) \in \chi$ denotes the resource and power allocation and the set χ is given by

$$\chi = \left\{ (\mathbf{x}, \mathbf{p}) \geq 0 : \sum_{i \in \mathcal{M}} x_i \leq 1, \sum_{i \in \mathcal{M}} l_i(t) \cdot p_i \leq I \right\} \quad (12)$$

$l_i(t)$ being the normalized interference of user i at time slot t . We use bold symbols to denote vectors, e.g., $\mathbf{x} = \{x_1, x_2, \dots, x_M\}$, $\mathbf{p} = \{p_1, p_2, \dots, p_M\}$. We adopt the gradient-based scheduling framework [11]–[15]. Specifically, the scheduler solves the following optimization

$$\max \sum_{i \in \mathcal{M}} \omega_i(t) r_i(t) \quad (13)$$

where $\omega_i(t) \geq 0$ is a time-varying QoS weight assigned to the i -th user at time t .

We concentrate on weights that depend on the average throughput attained by each user up to the t -th slot, and capture some fairness notation. For example, $\omega_i(t) = \frac{1}{T_i(t)}$, where $T_i(t)$ is the average throughput of user i at time t , which captures proportional fairness [11], [14], [15].

Note that (13) must be re-solved at each scheduling instant (e.g., each sub-frame) because of changes in both the resource

unit state as well as the weights.

For the ease of presentation, in the following we shall omit the time index t .

IV. JOINT UPLINK SCHEDULING AND POWER CONTROL

At the beginning of each time slot, the base station schedules a subset of the backlogged users to the available resource units and assigns transmission power to each scheduled user. The base station aims at maximizing the achievable rate while providing users with a fair share of resources (according to a predefined fairness metric) and maintaining a bounded interference with neighboring cells.

In conjunction with the scheduler and the power control, a rate adaptation mechanism adjusts the transmission rate according to the allocated power. Clearly, the allocated throughput in a time slot (sub-frame) to a scheduled user is derived from the allocated resource units and the allocated rate (i.e., power). In other words, the scheduling and power allocation are coupled, in devising the allocated throughput, and should be performed jointly. The power control scheme optimizes the trade-off between allocated rate and contribution to the overall Noise Rise. Typically, users that are far from the base station are required to transmit in a high power in order to maintain a reasonable rate. However, these users are closer to neighboring cells, hence contribute more to the noise rise in those cells. That is, on the other hand, cell edge users are required to transmit in a low power in order to bound the interference with neighboring cells.

At the beginning of each time slot the scheduling and power control scheme selects a feasible resource and power allocation tuple (\mathbf{x}, \mathbf{p}) that complies with the noise rise constraints and maximizes a time-varying weight assigned to each user, i.e.,

$$\max_{(\mathbf{x}, \mathbf{p}) \in \chi} \sum_{i \in \mathcal{M}} \omega_i \cdot r_i(x_i, p_i) \quad (14)$$

where $r_i(x_i, p_i)$ is the rate related to the resource and power allocation, and $\omega_i \geq 0$ is the time-varying weight assigned to the i -th user at the beginning of the time slot. These weights are the gradient of an increasing concave utility function of each user. Taking the rate as $r_i(x_i, p_i) = x_i \log \left(1 + \frac{p_i e_i}{x_i} \right)$ (11), we can formulate the joint power control and scheduling with noise rise constraint optimization problem.

For each time slot, find the channel allocated to each user, denoted by $\mathbf{x} = \{x_1, x_2, \dots, x_M\}$, as well as the set of power assigned to each user, denoted by $\mathbf{p} = \{p_1, p_2, \dots, p_M\}$, such that

$$\begin{aligned} & \max_{\mathbf{x}, \mathbf{p}} \sum_{i \in \mathcal{M}} \omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right) \\ & \text{subject to } x_i \geq 0, \forall i \in \mathcal{M}, \\ & p_i \geq 0, \forall i \in \mathcal{M}, \\ & \sum_{i \in \mathcal{M}} x_i \leq 1, \\ & \sum_{i \in \mathcal{M}} l_i p_i \leq I. \end{aligned} \quad (15)$$

Proposition 1. *Optimization problem (15) is convex with linear constraints.*

For a proof, see Appendix A.

From Proposition 1, it is clear that the optimal solution can be found numerically using standard optimization techniques. For example, it can be found through a similar method to that in [5], [16]. Yet, this direct approach might be prohibitively complex and infeasible for practical implementation in commercial base stations. Accordingly, a simpler solution is called for. In the next section, we give an efficient iterative algorithm, which uses the *analytical* solutions to two related sub-problems, to solve the above problem. Moreover, we show that the iterative algorithm converges to the global optimum.

V. AN ANALYTIC SOLUTION TO THE JOINT SCHEDULING AND POWER CONTROL PROBLEM

In this section we show that the optimal solution can be viewed as two *intertwined water-filling-like* problems, facilitating a highly efficient solution which solves the complete optimization problem by fixing a subset of the variables each time (either powers or bandwidth) and solving the resulting water-filling problem. We show that this iterative procedure is bound to converge, and, moreover, give analytical bounds on the possible values of the slack variables in each of the separate water filling problems, enabling us to converge to their solutions using a fast binary search.

The optimization problem we discuss is as follows.

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{p}}{\text{maximize}} && \sum_{i \in \mathcal{M}} \omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right) \\ & \text{subject to} && x_i \geq 0, \forall i \in \mathcal{M}, \\ & && p_i \geq 0, \forall i \in \mathcal{M}, \\ & && \sum_{i \in \mathcal{M}} x_i = 1, \\ & && \sum_{i \in \mathcal{M}} l_i p_i = I. \end{aligned} \quad (16)$$

Denote $\max\{x, 0\}$ by $[x]^+$. We first consider the analytical solution to this problem. Proposition 2 below gives a set of equations which the optimal bandwidth and power allocations satisfy.

Proposition 2. *Consider the joint power and bandwidth optimization problem in (16). The optimal power and bandwidth allocations, $\{p_i^*\}_{i \in \mathcal{M}}$, and $\{x_i^*\}_{i \in \mathcal{M}}$, respectively, satisfy*

$$p_i^* = x_i^* \left[\frac{\omega_i}{\lambda_1 l_i} - \frac{1}{e_i} \right]^+ \quad (17)$$

$$x_i^* = [\tilde{x}_i]^+ \quad (18)$$

where \tilde{x}_i is the solution to the following equation

$$\omega_i \log \left(1 + \frac{p_i^* e_i}{\tilde{x}_i} \right) - \frac{p_i^* e_i \omega_i}{\tilde{x}_i + p_i^* e_i} + \lambda_2 = 0 \quad (19)$$

and λ_1, λ_2 are set such that

$$\sum_i l_i x_i^* \left[\frac{\omega_i}{\lambda_1 l_i} - \frac{1}{e_i} \right]^+ = I. \quad (20)$$

$$\sum_i x_i^* = 1. \quad (21)$$

For a proof, see Appendix B

A. An Iterative Algorithm

While Proposition 2 gives necessary and sufficient conditions for power allocations \mathbf{p} and bandwidth allocations \mathbf{x} to be optimal, its direct computation is cumbersome, as the equations for both types of variables are intertwined. However, in this sub-section, we show that the problem in (16) can be solved optimally by a highly efficient iterative algorithm, which, unlike standard iterative optimization procedures, does not jointly optimize on all variables, but rather utilizes the fact that when separating the power variables from the bandwidth ones, each optimization problem has a relatively easy water-filling-like analytical solution.

The important observation is as follows. Fixing the bandwidth variables $\{x_i\}_{i \in \mathcal{M}}$, the resulting optimization problem is

$$\begin{aligned} & \underset{\mathbf{p}}{\text{maximize}} && \sum_{i \in \mathcal{M}} \omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right) \\ & \text{subject to} && p_i \geq 0, \forall i \in \mathcal{M}, \\ & && \sum_{i \in \mathcal{M}} l_i p_i = I. \end{aligned} \quad (22)$$

The solution to this problem is the well-known *water filling*, e.g., [17, Example 5.2]. Hence, it is easily solvable (note that the weights $\omega_i x_i$ and the noise rise constraints l_i only serve as scaling factors, and do not change the essence of the problem in the *separated case*). Fixing the power variables $\{p_i\}_{i \in \mathcal{M}}$ results in a relatively similar optimization problem, which although involves an implicit equation on each x_i , is also straightforward to solve. The iterative algorithm will then alternate between the two solutions, fixing one set of variables based on the results of the previous iteration. A pseudo code of the algorithm follows.

```

ITERATIVE-WATER-FILLING( $\mathbf{e}, \mathbf{l}, I$ )
1   $\mathbf{x} \leftarrow \mathbf{x}^0$  : such that  $x_i^0 > 0 \forall i, \sum_i x_i^0 = 1$ 
2  repeat
3       $\lambda_1 \leftarrow \text{SOLVE}(\sum_i l_i x_i \left[ \frac{\omega_i}{\lambda_1 l_i} - \frac{1}{e_i} \right]^+ = I)$ 
4       $p_i \leftarrow x_i \left[ \frac{\omega_i}{\lambda_1 l_i} - \frac{1}{e_i} \right]^+ \forall i$ 
5       $\lambda_2, \mathbf{x} \leftarrow \text{SOLVE}(\sum_i x_i = 1,$ 
6           $\omega_i \log \left( 1 + \frac{p_i e_i}{x_i} \right) - \frac{p_i e_i \omega_i}{x_i + p_i e_i} + \lambda_2 = 0)$ 
7      until CONVERGE
8  return  $\{\mathbf{x}, \mathbf{p}\}$ 

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When evaluating Algorithm ITERATIVE-WATER-FILLING, the two key aspects are complexity and convergence. First, consider the number of operations in each iteration. The first step,

as mentioned, is a basic water-filling procedure. The value of λ_1 can be calculated by first ordering the users according to $\frac{l_i}{\omega_i e_i}$, then including users from the smallest to the largest until the “water level” $\frac{1}{\lambda_1}$ satisfies the noise rise constraint. This is done in $O(M \log M)$. As for the second step, the solution is more involved, since it cannot be solved explicitly. However, as the following proposition states, the solution is monotonic in λ_2 , with *upper and lower bounds* on the value of the optimal λ_2 , hence can be solved efficiently by a logarithmic time binary search.

Proposition 3. *For each $i \in \mathcal{M}$, let \tilde{x}_i be the solutions to*

$$\omega_i \log \left(1 + \frac{p_i e_i}{x_i} \right) - \frac{p_i e_i \omega_i}{x_i + p_i e_i} + \lambda_2 = 0. \quad (23)$$

Then, for all $i \in \mathcal{M}$, every $\lambda_2 \leq 0$ and any $\omega_i \geq 0, p_i \geq 0$ and $e_i \geq 0$, we have:

- 1) $\tilde{x}_i \geq 0$ and $\sum_{i \in \mathcal{M}} \tilde{x}_i$ is monotonically increasing in λ_2 .
- 2) The value of λ_2 in (23) such that $\sum_i \tilde{x}_i = 1$ satisfies $\lambda_2^{\min} \leq \lambda_2 \leq \lambda_2^{\max}$, where

$$\begin{aligned} \lambda_2^{\min} &= \min_{i \in \mathcal{M}} \left\{ \omega_i \left(\frac{M p_i e_i}{1 + M p_i e_i} - \log(1 + M p_i e_i) \right) \right\} \\ \lambda_2^{\max} &= \max_{i \in \mathcal{M}} \left\{ \omega_i \left(\frac{p_i e_i}{1 + p_i e_i} - \log(1 + p_i e_i) \right) \right\} \end{aligned}$$

Thus, when solving the bandwidth iteration in Algorithm ITERATIVE-WATER-FILLING, simply compute λ_2^{\min} and λ_2^{\max} , and apply a binary search for the value of λ_2 such that $|\sum_i x_i - 1| \leq \epsilon$. The computational cost is $O(M \log(1/\epsilon))$ assuming (23) is solved for x_i in $O(1)$ for fixed λ_2 .

For a proof, see Appendix C.

Finally, we mention two important results on the convergence of Algorithm ITERATIVE-WATER-FILLING. As mentioned, the algorithm iteratively solves two optimization problems, each one involving half of the parameters to be optimized (either powers or bandwidth allocations). This is an alternating optimization procedure. While this procedure can fail for some utility functions¹, it is important to note that in the specific case of our joint power and bandwidth scheduling with noise-rise constraint, it is bound to converge.

Corollary 1. *Assume Algorithm ITERATIVE-WATER-FILLING is used to solve (16). Then the power assignments \mathbf{p} and channel allocations \mathbf{x} converge to the global optimum of (16).*

For a proof, see Appendix D.

Note that the result in [18, Theorem 2] implies that as long as the power and bandwidth allocations \mathbf{p}, \mathbf{x} are not in the range of the *fixed points* of $T[\mathbf{p}, \mathbf{x}]$ (an iteration of the algorithm), the negative utility function is strictly decreasing in each iteration. Moreover, the result in Corollary 1 can be made even stronger if the negative utility function is *strictly*

convex in the range. Consider, for example, the two user case. While the function

$$-\omega_1 x_1 \log \left(1 + \frac{p_1 e_1}{x_1} \right) - \omega_2 x_2 \log \left(1 + \frac{p_2 e_2}{x_2} \right)$$

is not strictly convex for all p_1, p_2, x_1, x_2 (the Hessian matrix is not of full rank and hence not positive definite), it is *strictly convex* under the constraints $x_1 + x_2 = 1$ and $l_1 p_1 + l_2 p_2 = I$ (for positive x_i and p_i). In this case, by [18, Theorem 3], the alternating optimization in Algorithm ITERATIVE-WATER-FILLING converges *q-linearly* to the global optimum, from any starting point $(p_1, p_2, x_1, x_2)^0$ in the range. That is, each iteration of the algorithm decreases the distance to the global optimum by a constant multiplicative factor $q \in [0, 1)$. In other words,

$$\|(\mathbf{p}, \mathbf{x})_t - (\mathbf{p}^*, \mathbf{x}^*)\| \leq q \|(\mathbf{p}, \mathbf{x})_{t-1} - (\mathbf{p}^*, \mathbf{x}^*)\|.$$

VI. CONSTRAINED NOISE RISE DENSITY

In this section we consider a second noise rise settings where the *noise rise density* is constrained. That is, instead of noise rise budget over the whole frequency band, the noise rise per resource unit is constrained.

The constrained noise rise density approach is also suitable to the contiguous resource allocation scheme (i.e., without any sub-band partitioning and permutations). It is also viable to the LTE SC-FDMA uplink allocation.

Formally, the constraints on the noise rise density for each allocated user i is given by

$$l_i \cdot \frac{p_i}{x_i} \leq I \quad \forall i. \quad (24)$$

It is easy to see, by summing both sides of (24) over all $i \in \mathcal{M}$, that a resource and power allocation $\{\mathbf{x}, \mathbf{p}\}$ that complies with (24) will not exceed the noise rise constraint over the whole frequency band.

The scheduling and power control problem with constrained noise rise density is as follows.

$$\begin{aligned} &\underset{\mathbf{x}, \mathbf{p}}{\text{maximize}} && \sum_{i \in \mathcal{M}} \omega_i(t) x_i \log \left(1 + \frac{p_i e_i}{x_i} \right) \\ &\text{subject to} && x_i \geq 0, \quad \forall i \in \mathcal{M}, \\ &&& p_i \geq 0, \quad \forall i \in \mathcal{M}, \\ &&& \sum_{i \in \mathcal{M}} x_i \leq 1, \\ &&& l_i \cdot \frac{p_i}{x_i} \leq I, \quad \forall i : x_i > 0. \end{aligned} \quad (25)$$

It is easy to see that bounding the noise rise density allows us to decouple the scheduling from the power allocation. That is, the power allocation and (consequently) rate assignment are determined by the required noise rise density regardless of the allocated resource units. Accordingly, for each user i the allocated power density (e.g., power per sub-carrier) is given by $\frac{p_i}{x_i} = \frac{I}{l_i}$, and the achievable rate of user i is given by $x_i \log \left(1 + \frac{I e_i}{l_i} \right)$. Now, let the user i^* be such

¹E.g., $x^2 - 3xy + y^2$ when alternating between the optimization on y and the optimization on x

that $i^* = \arg \max_i \left\{ \omega_i(t) \log \left(1 + \frac{I_{e_i}}{l_i} \right) \right\}$. The power and frequency allocation that solves (25), allocates all frequency band to user i^* with the power $p_{i^*} = \frac{I}{l_{i^*}}$. A pseudo code that depicts the algorithm follows.

```

RESOURCE-ALLOCATION( $t, \mathbf{L}, \mathbf{SIR}^{DL}$ )
1   $\mathbf{x} \leftarrow \mathbf{0}, \mathbf{p} \leftarrow \mathbf{0}$ 
2  for  $i \in \mathcal{M}$ 
3  do  $\omega_i \leftarrow \text{FAIRNESS-WEIGHT}(t)$ 
4      $l_i \leftarrow \frac{L_i}{\text{SIR}_i^{DL}}$ 
5      $\frac{p_i}{x_i} \leftarrow \frac{I}{l_i}$ 
6      $r_i \leftarrow \text{RATE-ADAPTATION}(\frac{p_i}{x_i})$ 
7   $i^* \leftarrow \arg \max_{i \in \mathcal{M}} \{ \omega_i \cdot r_i \}$ 
8   $x_{i^*} \leftarrow 1$ 
9   $p_{i^*} \leftarrow \frac{I}{l_{i^*}}$ 
10 return  $\{\mathbf{x}, \mathbf{p}\}$ 

```

$\mathbf{L} = \{L_1, L_2, \dots, L_M\}$, where L_i is the channel gain between user i and its serving base station. $\mathbf{SIR}^{DL} = \{\text{SIR}_1^{DL}, \text{SIR}_2^{DL}, \dots, \text{SIR}_M^{DL}\}$, where SIR_i^{DL} is the down-link signal-to-interference ratio at the receiver of user i . l_i is the user i normalized interference. ω_i is obtained through $\text{FAIRNESS-WEIGHT}(t)$, which computes the user i fairness weight at time t . Typically, $\omega_i = \frac{1}{T_i(t)}$, where $T_i(t)$ is the average throughput of user i at time t .

Note that, the algorithm RESOURCE-ALLOCATION does not necessarily assume a Gaussian Multiple Access Channel (i.e., $r_i = \log \left(1 + \frac{p_i e_i}{x_i} \right)$). Instead, the algorithm calls a rate adaptation mechanism to capture the achievable rate. It is made possible, since our assumption on bounded noise rise density allows the computation of the uplink transmission power density regardless of the schedule. The transmission power density is sufficient for the rate adaptation. Clearly, in practice, a dynamic rate adaptation mechanism would better estimate the achievable rate and facilitate a better utilization of the multi-user diversity resulting in a higher throughput.

VII. SIMULATION RESULTS

A. Numerical Results - Shannon Capacity

In this sub-section, we give the numerical results for the algorithms in section V and section VI, and compare them to a fixed power scheme. Our main figure of merit is the *total throughput* in the system, summed over all frames and all cells. However, to stress out the benefits of the noise-rise based schemes, the standard deviation of the interference seen at the cells as well as the actual power used are also given. In particular, the following simulation was carried out.

The considered deployment include 72 hexagon-shaped cells (base stations), with 722 users (mobile stations). The user locations were drawn uniformly at random on the space, with the restriction of a minimum of 2 users per cell. The topology thus created is given in Figure 1. The path losses (in dB) were calculated according to the COST-Hata model [19,

Chapter 4, equation 4.4.3]

$$PL = 46.3 + 33.9 \log_{10}(f) - 13.82 \log_{10}(h_B) - ah_m \\ + (44.9 - 6.55 \log_{10}(h_b)) \log_{10}(d) + c_m,$$

where $f = 2000 \text{ MHz}$ was the transmission frequency, $h_B = 50$ was the base station antenna effective height, ah_m was the mobile station antenna height correction factor and c_m was taken as 0 for a medium size cite. The simulation included 80 frames, with the topology and path gains fixed throughout.

The algorithm used to solve the optimal, noise-rise constraint problem (15) was Algorithm ITERATIVE-WATER-FILLING. In the noise-rise density constraint algorithm only a single user per cell is scheduled according to Algorithm RESOURCE-ALLOCATION. That is, this user is selected according to the maximal value of $\omega_i x_i \log \left(1 + \frac{I_{e_i}}{l_i x_i} \right)$, assuming the user exhausts the noise-rise budget on its own. In the fixed power scheme, again, only a single user per cell is scheduled. However, in this schemes, the allocated power is fixed at the same constant P for all users, and only the user which under such an allocation will have the maximal normalized rate $\omega_i \log \left(1 + \frac{P e_i}{x_i} \right)$ is scheduled. Of course, for a fair comparison, the fixed power P is set such that the mean interference caused by all schemes is the same.

The weights ω_i were initialized to a constant value, and were updated after each frame according to [11]

$$T_i(t) = T_i(t-1) + (1 - \beta) \cdot B_i(t-1)$$

and

$$\omega_i(t) = 1/T_i(t),$$

with $\beta = 0.9$ and $B_i(t-1)$ is the number of bits delivered to user i at sub-frame $t-1$ (0 if user i was not scheduled at frame $t-1$).

In all three schemes, the actual throughputs after each frame were calculated according to the true SINR at the receivers (Shannon capacity for band-limited Gaussian channel). This is to simulate the exact scenario for which our analytic claims apply. Note, however, that this is fundamentally different from the all-encompassing simulation described in the next section, where specific modulations and packet loss rates are taken into account.

The results are given in Figures 2 to 4. In Figure 2, the total throughput, over all cells and frames is given. The benefit of both noise-rise schemes over the fixed power scheme is clear for all noise-rise values (equivalently, all SNR values). The benefit of the optimal algorithm over the sub-optimal is also clear, as it indeed usually schedules more than one user per cell. Note that it is possible, though, to construct topologies in which the benefit is small.

A key advantage in the noise-rise schemes is that the interference seen at the neighboring cells is concentrated around the fixed noise-rise value I , allowing them to plan modulation and coding accordingly. Clearly, when the variance of the actual interference observed at the receivers is high, it is harder to choose the appropriate modulation and coding, forcing the senders to either aim at lower modulation and

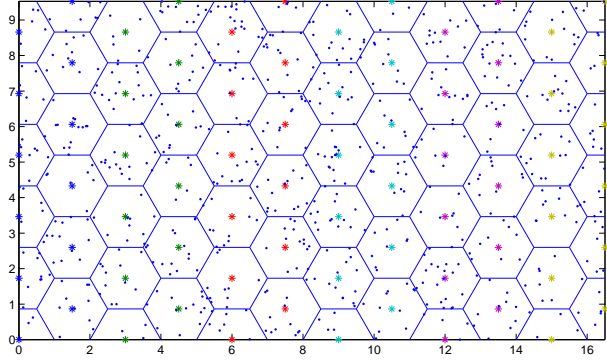


Fig. 1. The network topology. 72 cells with 722 mobile stations, distributed uniformly at random. Each cell was restricted to have a minimum of 2 mobile stations.

coding schemes or suffer a high packet loss. The variance of the observed interference for all schemes is given in Figure 3. Note that the average is fixed to the same value in all schemes.

Finally, we address the lack of maximum power constraint (head room). The four figures in Figure 4 depict the histograms of the actual powers allocated to the users in the optimal noise-rise scheme. It is clear that a substantial fraction of the users is allocated powers *below or around* that allocated by the fixed power scheme, and even the small fraction with high powers does not require a consequential increase in power. Thus, although the optimal scheme is not head room constrained, its actual power usage is moderate.

B. An IMT-Advanced Simulation

We illustrate the advantages of the noise rise approach via extensive system simulations. To that end, we developed a Matlab simulation that comply with the guidelines for evaluation of radio interface technologies for IMT-Advanced [8]. Specifically, we simulate a two tier hexagonal deployment with 19 sites, each contain three sectors. Each sector operates as an independent base station. Users are located uniformly across the deployment with 10 users in each cell. Statistics are collected from all cells, which is facilitated by a wrap around geometry such that all cells suffer from interference from two tiers neighboring cells.

Following the IMT-Advanced guidelines, we implemented the WINNER II [20] stochastic channel modeling. The modeling approach is based on the geometry of the network layout. The large-scale parameters such as path loss and shadowing factor are generated according to the geometric positions of the base station and the user. Then the statistical channel behavior is defined by distribution functions of delay and angle as well as by the power delay and angular profiles. We consider the urban macro-cell scenario, where typically mobile stations are

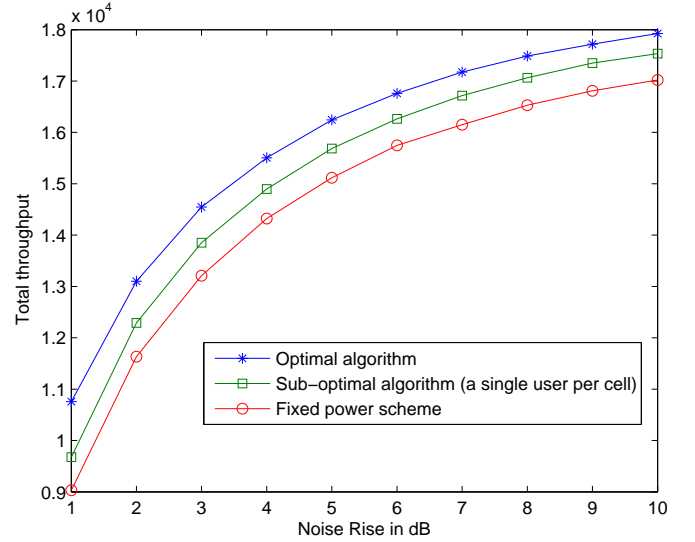


Fig. 2. The commulative throughput over all frames (80) and all cells. The optimal algorithm of section V is compared to the sub-optimal algorithm given in section VI and the fixed power scheme.

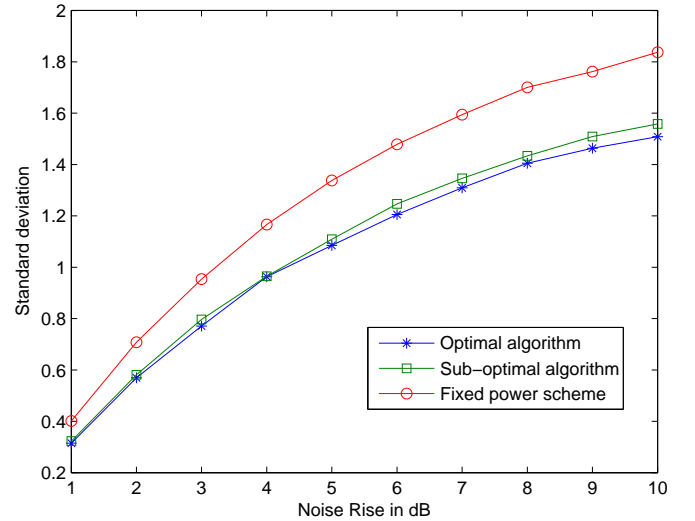
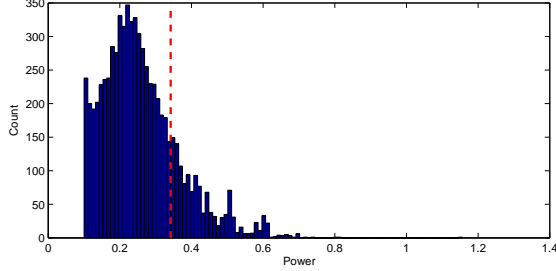
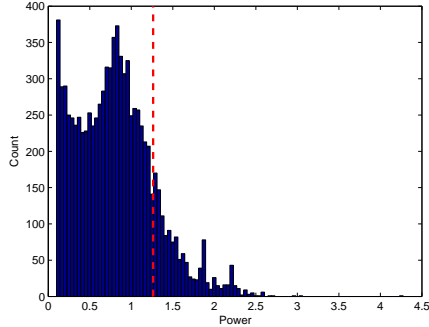


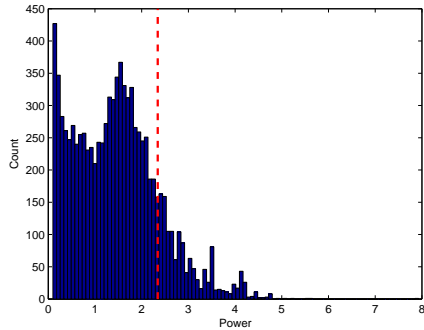
Fig. 3. Standard deviation of the interference seen at the cells (over all frames and all cells). The optimal algorithm of section V is compared to the sub-optimal algorithm given in section VI and the fixed power scheme. While the average interference is constrained to the same value in all schemes, the two noise-rise schemes (the optimal and the sub-optimal algorithms) exhibit a significantly lower standard deviation. That is, the interference seen under these schemes is centered around the average value, allowing better rate planning by the base stations.



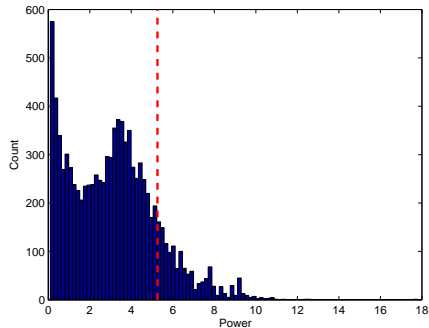
(a) 2 dB Noise rise



(b) 5 dB Noise rise



(c) 7 dB Noise rise



(d) 10 dB Noise rise

Fig. 4. Histograms of the allocated powers in the optimal algorithm (only powers above 0.1 are included). The dashed red line represents the power allocated by the fixed power scheme. It is clear that the number of users with high powers compared to the fixed power scheme is small. Moreover, even those users do not use more than double or three times as much power compared to the fixed power scheme.

located outdoors at street level and base stations are fixed clearly above surrounding building heights. The assumed inter-site distance is $500m$. We assume two receive antennas at the base station and one transmit antenna and the mobile station. The maximal uplink transmit power is $24dBm$. The user and the base station antenna gain is 0 and $17dBi$, respectively. The assumed noise figure at the user terminal and the base station is 7 and $5dB$, respectively.

We consider the IEEE 802.16m uplink frame structure (*Time Domain Duplexing - TDD* mode). The assume total bandwidth is of $10MHz$, which is occupied by 48 resource units. The simulation runs sub-frame by sub-frame, performing the scheduling and power allocation at the beginning of each sub-frame. We run the simulation over 1000 sub-frames for 5 drops. In each drop all users are reallocated across the deployment.

The *exponential-effective SINR Mapping (EESM)* approach [21] is used to map the system level SINR onto the link level curves to determine the resulting block error rate. Upon block errors, a synchronized retransmission is scheduled with the highest priority after 5 sub-frames from the original transmission. Upon reception of retransmission the receiver performs chase combining. The maximal number of retransmissions is 4.

We assume a full-queue model where all users have backlogged traffic. The weights ω_i were initialized to a constant value, and were updated after each frame according to $T_i(t) = T_i(t-1) + (1-\beta) \cdot B_i(t-1)$ and $\omega_i(t) = 1/T_i(t)$, with various values of β .

Figure 5 compares three power control schemes, namely, (i) constrained noise rise (termed N.R) based on Algorithm ITERATIVE-WATER-FILLING; (ii) constrained noise rise density (termed N.R. density) based on Algorithm RESOURCE-ALLOCATION; and (iii) target received SINR (termed SINR). The last scheme represents the traditional power allocation scheme (which aims at obtaining the required SINR level at the receiver) and is used as a benchmark for our noise rise approach. One can see that both noise rise algorithms obtain a dramatically better spectral efficiency. For example, the constrained noise rise approach obtains spectral efficiency of 1 [bit/Hz/Sec] for $\beta = 0.99$ where the SINR based scheme obtains only 0.5 [bit/Hz/Sec]. Yet, the traditional approach obtains a slightly higher cell edge spectral efficiency. Interestingly, the constrained noise rise density algorithm obtains a better spectral efficiency than the original scheme. This advantage is due to the difference between the Gaussian Multiple Access Channel model and the simulated IMT advanced EESM channel model, which does not disrupt the constrained noise rise density approach (as discussed in section VI).

VIII. CONCLUSION

In this paper, we considered a joint scheduling and power allocation problem. Specifically, to mitigate inter-cell interference, we suggested a novel approach, which considers the interference caused by a base station to neighboring base stations as a resource to be allocated, similar to bandwidth

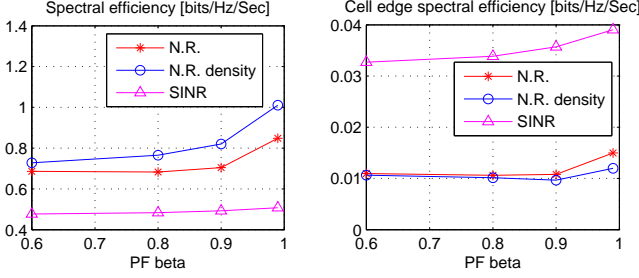


Fig. 5. Uplink spectral efficiency (a) average (b) cell edge.

or power. The essence of this approach is as follows. Each user, based on its channel gains, creates a different interference to neighboring cells when transmitting. A base station, while allocating power and bandwidth to its subscribers, does so in a way such that the total aggregated interference its users generate does not pass a certain value - its noise rise budget.

We rigorously formulated the problem as a convex optimization problem with linear constraints and suggested an efficient iterative algorithm for its solution, based on known and new water-filling based solutions to its separate problems. We then devised a modified algorithm, which optimizes power and bandwidth allocations assuming the noise rise is constraint for each sub-channel (a portion of the bandwidth allocated to a single user). This algorithm is highly efficient, yet performs better than known techniques, approximating the performance of the optimal algorithm. These performance guarantees were also depicted in two extensive simulations, one based on the analytical expressions and one based on practical system built based on the IMT-Advanced specification.

Potential applications of the suggested algorithms include, but are not limited to, uplink scheduling in IEEE 802.16e/m, 3GPP LTE, and more.

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APPENDIX A

PROOF OF PROPOSITION 1

Proof: Consider the negative utility function

$$-\sum_{i \in \mathcal{M}} \omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right). \quad (26)$$

We first show that this function is convex. To this end, consider a single summand

$$f(x_i, p_i) = -\omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right).$$

The Hessian matrix, restricted to the variables x_i and p_i is given by

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_i^2} & \frac{\partial^2 f}{\partial x_i \partial p_i} \\ \frac{\partial^2 f}{\partial p_i \partial x_i} & \frac{\partial^2 f}{\partial p_i^2} \end{pmatrix} = \frac{\omega_i e_i^2}{(1 + \frac{e_i p_i}{x_i})^2 x_i} \begin{pmatrix} \frac{p_i^2}{x_i^2} & -\frac{p_i}{x_i} \\ -\frac{p_i}{x_i} & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} \frac{p_i^2}{x_i^2} & -\frac{p_i}{x_i} \\ -\frac{p_i}{x_i} & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \left(\alpha_1 \frac{p_i}{x_i} - \alpha_2 \right)^2$$

Hence, the Hessian matrix of (26), $H(\mathbf{x}, \mathbf{p})$, satisfies

$$\begin{aligned} & (\alpha_1, \dots, \alpha_{2M}) H(\mathbf{x}, \mathbf{p}) (\alpha_1, \dots, \alpha_{2M})^T \\ &= \sum_{i \in \mathcal{M}} \frac{\omega_i e_i^2}{(1 + \frac{e_i p_i}{x_i})^2 x_i} \left(\alpha_{2i-1} \frac{p_i}{x_i} - \alpha_{2i} \right)^2 \geq 0 \end{aligned} \quad (27)$$

for all $(\alpha_1, \dots, \alpha_{2M}) \in \mathbb{R}^{2M}$. ■

APPENDIX B PROOF OF PROPOSITION 2

Proof: In Appendix A we show the convexity of the optimization problem. The proposition will now follow from a straight forward application of the KKT conditions [17, Section 5.5.3]. Namely, we write

$$\begin{aligned} L(\mathbf{p}, \mathbf{x}, \lambda_1, \lambda_2, \{\mu_i\}_{i \in \mathcal{M}}, \{\tilde{\mu}_i\}_{i \in \mathcal{M}}) = \\ \sum_{i \in \mathcal{M}} \omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right) + \lambda_1 \left(\sum_{i \in \mathcal{M}} l_i p_i - I \right) \\ + \lambda_2 \left(\sum_{i \in \mathcal{M}} x_i - 1 \right) - \sum_{i \in \mathcal{M}} \mu_i x_i - \sum_{i \in \mathcal{M}} \tilde{\mu}_i p_i \quad (28) \end{aligned}$$

and the proposition follows by requiring $\nabla L = 0$ and that for all $i \in \mathcal{M}$ we have

$$\tilde{\mu}_i p_i = 0, \quad \mu_i x_i = 0, \quad \tilde{\mu}_i \geq 0, \quad \mu_i \geq 0.$$

■

APPENDIX C PROOF OF PROPOSITION 3

Proof: To prove item 1) we proceed as follows. Set $\frac{p_i e_i}{x_i} = \alpha$ and consider the function $f(\alpha) = \log(1 + \alpha) - \frac{\alpha}{1 + \alpha}$. It easy to verify that $f(0) = 0$ and that $f(\alpha)$ is non-negative and monotonically increasing for any $\alpha \geq 0$. Thus, for any $\frac{\lambda_2}{\omega_i} < 0$, the equation $f(\alpha) = -\frac{\lambda_2}{\omega_i}$ will have a unique solution at some $\alpha = \frac{p_i e_i}{x_i} > 0$. Hence, $\tilde{x}_i \geq 0$. Moreover, for fixed p_i, e_i, ω_i , the smaller λ_2 is, the larger is the solution to $f(\alpha) = -\frac{\lambda_2}{\omega_i}$, that is, the smaller \tilde{x}_i , for all i , and hence the smaller is $\sum_i \tilde{x}_i$.

To prove item 2) we note the following. Since $\sum_i \tilde{x}_i$ is monotonically increasing in λ_2 , the value of λ_2 such that $\sum_i \tilde{x}_i = 1$ is clearly upper bounded by the value of λ_2 for which the “weakest” user, the user which requires the largest λ_2 in order to achieve $\tilde{x}_i = 1$, indeed gets it (since in this case all other users will have $\tilde{x}_{i'} > 1$ and $\sum_i \tilde{x}_i$ will clearly surpass 1). Thus, the optimal λ_2 is at most

$$\begin{aligned} \max_{i \in \mathcal{M}} \{ \lambda_2 \quad \text{s.t.} \quad \tilde{x}_i = 1 \} = \\ \max_{i \in \mathcal{M}} \left\{ \omega_i \left(\frac{p_i e_i}{1 + p_i e_i} - \log(1 + p_i e_i) \right) \right\}. \quad (29) \end{aligned}$$

On the other hand, the value of λ_2 such that $\sum_i \tilde{x}_i = 1$ is clearly lower bounded by the value of λ_2 for which the “strongest” user, the user which requires the minimal λ_2 in order to have $\tilde{x}_i = 1/M$, indeed gets it (since in this case all other users will have $\tilde{x}_{i'} < 1/M$ and thus $\sum_i \tilde{x}_i$ will be strictly smaller than 1). As a result, the optimal λ_2 is at least

$$\begin{aligned} \min_{i \in \mathcal{M}} \{ \lambda_2 \quad \text{s.t.} \quad \tilde{x}_i = 1/M \} = \\ \min_{i \in \mathcal{M}} \left\{ \omega_i \left(\frac{M p_i e_i}{1 + M p_i e_i} - \log(1 + M p_i e_i) \right) \right\}. \quad (30) \end{aligned}$$

This completes the proof. ■

APPENDIX D PROOF OF COROLLARY 1

Proof: We first show that the negative utility function $-\sum_{i \in \mathcal{M}} \omega_i x_i \log \left(1 + \frac{p_i e_i}{x_i} \right)$ satisfied an existence and uniqueness constraint [18, Section 2]. That is, fixing any $2M - 1$ variables and optimizing on the remaining variable, the resulting problem has a unique (global) minimizer in the range. This is easily seen from the Hessian matrix calculated in the proof of Proposition 2, as the negative utility function is convex in each of the variables.

We now mention that the transform $(\mathbf{p}, \mathbf{x})_t = \mathbf{T}[(\mathbf{p}, \mathbf{x})_{t-1}]$ defined by one iteration of Algorithm ITERATIVE-WATER-FILLING has no fixed points (for which $T[\mathbf{v}] = \mathbf{v}$) besides the global optimum of (16). This is, again, since the negative utility function is convex. The corollary will now follow by applying [18, Theorem 2]. ■